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Consecutive Residues or Non-Residues in the Gaussian Integers

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This paper considers the following questions.

(i) Are there always m "consecutive" kth power residues modulo a Gaussian prime within a certain distance of the origin with the distance independent of the Gaussian prime?

(ii) Are there always m "consecutive" elements in some coset with respect to the multiplicative subgroup of kth power residues modulo a Gaussian prime that are within a certain distance of the origin with the distance of the Gaussian prime?

The answers are shown to be *yes* if $m = 2, 3$ and $k = 2, 3$ and *no* if $k \geq 2$ and $m \geq 4$. The problems are not solved for $k > 3$ and $m = 2, 3$.

INTRODUCTION

In 1928 A. Brauer [1] showed that for any positive integers m and k there is a M such that for each prime $p > M$ there is an a such that a kth power Dirichlet character modulo p has the property

$$\chi(a+1) = \chi(a+2) = \ldots = \chi(a+m) = 1. \tag{I}$$

In 1963 D. H. Lehmer, E. Lehmer, and W. Mills [6] defined $\Lambda(k, m)$ to be the least upper bound such that for all but a finite number of primes there is an $a \leq \Lambda(k, m)$ which has property (I) with a kth power Dirichlet character modulo that prime.

Someone noticed that

$$\Lambda(2,2) = 9 \quad \text{except for} \quad p = 2,\ 3, \text{ and } 5.$$

M. Dunton [2] established that $\Lambda(3, 2) = 77$ except for $p = 2$, 7, and 13.

W. Mills and R. Bierstedt [10] showed that $\Lambda(4, 2) = 1224$ except for $p = 2$, 3, 5, 13, 17, 41.

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D. H. Lehmer, E. Lehmer, and W. H. Mills [6] proved $\Lambda(5,2) = 7888$ except $p = 2$, 11, 41, 71, 101 and $\Lambda(6,2) = 202124$ except $p = 2$, 3, 5, 7, 13, 19, 43, 61, 97, 157, 277.

D. H. Lehmer and Emma Lehmer [5] showed $\Lambda(2,3) = \infty$. D. H. Lehmer, Emma Lehmer, W. H. Mills, and J. Selfridge [7] proved that

$$\Lambda(3,3) = 23532 \quad \text{except for} \quad p = 2, 3, 7, 13, 19, 31, 37, 43, 61, 67, 79, 127, \text{and } 283.$$

D. H. Lehmer, Emma Lehmer, and J. Brillhart [8] showed that

$$\Lambda(7,2) \leq 1649375 \text{ with some exceptional primes.}$$

R. Graham [3] established that

$$\Lambda(k,m) = \infty \quad \text{for} \quad m \geq 4.$$

The author [4] defined $\Lambda^*(k,m)$ as the least bound such that for all but a finite number of primes there is a $b \leq \Lambda^*(k,m)$ such that a kth power Dirichlet character has the property that

$$\chi(b+1) = \chi(b+2) = \ldots \chi(b+m).$$

The results established were

(i) $\Lambda^*(2,2) = 3$, except 2

(ii) $\Lambda^*(3,2) = 8$, except 2

(iii) $\Lambda^*(4,2) = 20$, except $2, 3, 5$

(iv) $\Lambda^*(5,2) = 44$, except 2

(v) $\Lambda^*(6,2) = 80$, except $2, 3, 7$

(vi) $\Lambda^*(7,2) = 343$, except 2

(vii) $\Lambda^*(2,3) = \infty$

(viii) $\Lambda^*(k,m) = \infty \quad \text{for} \quad m \geq 4$.

It is the purpose of this paper to consider analogous problems in Gaussian integers.

Definitions and Results

Two Gaussian integers α and β will be called consecutive if $|\alpha - \beta| = 1$. A set of m Gaussian integers $\{\alpha_j\}_1^m$ will be called consecutive if $|\alpha_1 - \alpha_2| = 1$ and $\alpha_1 - \alpha_2 = \alpha_j - \alpha_{j+1}$, for $j = 1, 2, 3, \ldots, m-1$.

Let $\Lambda_G(k,m)$ be the least positive number such that in a circle of radius $\Lambda_G(k,m)$ centered at the origin there is for all but a finite number of Gaussian primes a set of m consecutive Gaussian integers, $\{\alpha_j\}_1^m$ with the property that $\chi(\alpha_j) = 1$ for a kth power Dirichlet character modulo that Gaussian Prime.

Let $\Lambda_G^*(k, m)$ be the least positive number such that in a circle of radius $\Lambda_G^*(k, m)$ centered at the origin there is for all but a finite number of Gaussian primes a set of m consecutive Gaussian integers, $\{\alpha_j\}_1^m$ with the property that $\chi(\alpha_j) = \chi(\alpha_1)$ for a kth power Dirichlet character modulo that Gaussian prime.

The results of this paper are:

A. $\Lambda_G^*(2,2) = 2$, except $1+i$

B. $\Lambda_G(2,2) = 2$, except $1+i$

C. $\Lambda_G^*(2,3) = \sqrt{18}$, except $1+i$, $2+i$, $2-i$, and 3

D. $\Lambda_G(2,3) = \sqrt{34}$, except $1+i$, $2+i$, $2-i$, 3, $4+i$, and $4-i$

E. $\Lambda_G^*(3,2) = \sqrt{8}$, except $1+i$

F. $\Lambda_G(3,2) = 15$, except $1+i$, $3+2i$, and $3-2i$

G. $\Lambda_G^*(k, m) = \infty$, $m \geq 4$

H. $\Lambda_G(k, m) = \infty$, $m \geq 4$.

Proofs

The proofs of A, B, C, D, E, and F are given in tabular form. Each table establishes an upper bound for the appropriate Λ or Λ^* by considering all possible values of χ for a set of Gaussian primes. A dash "—" in the column means that particular result is independent of that prime or unit. Some arguments are only previous arguments where the domain is changed from $a+bi$ to $a-bi$ for all pertinent entries. This will be noted by the abbreviation "Conj. arg." Following the table the exceptional Gaussian primes will be listed.

A generalization of a result of W. H. Mills [9] would guarantee that there are infinitely many Gaussian primes with the property that one of their kth power characters agrees with the values assigned to the primes smaller than the established bound. This establishes that the bound established in the table is best possible.

In Tables A, B, C, and D, the symbol χ is the quadratic character.

TABLE A

	i	$1+i$	Consecutives
1.	1	—	$\chi(i) = \chi(2i)$
2.	—	1	$\chi(1) = \chi(1+i)$
3.	-1	-1	$\chi(i) = \chi(1+i)$

Hence $\Lambda_G^*(2,2) \leq 2$. The prime 3 shows that line 2 in Table A gives the best possible result. The only exceptional prime is $1+i$.

TABLE B

	i	$1+i$	Consecutives
1.	1	—	$\chi(i) = \chi(2i)$
2.	—	1	$\chi(1) = \chi(1+i)$
3.	-1	-1	$\chi(1) = \chi(1-i)$

Hence $\Lambda_G(2,2) \leq 2$. The prime 3 shows that line 2 in Table B gives the best possible result. The only exceptional prime is $1+i$.

TABLE C

	i	$1+i$	$2+i$	$2-i$	3	$3+2i$	Consecutives
1.	1	1	—	—	—	—	$\chi(1-i) = \chi(1) = \chi(1+i)$
2.	1	—	—	—	1	—	$\chi(1) = \chi(2) = \chi(3)$
3.	1	-1	1	1	—	—	$\chi(2-i) = \chi(2) = \chi(2+1)$
4.	1	-1	1	-1	—	—	$\chi(1-i) = \chi(2-i) = \chi(3-i)$
5.	1	-1	-1	-1	-1	1	$\chi(3+i) = \chi(3+2i) = \chi(3+3i)$
6.	1	-1	-1	-1	-1	-1	$\chi(1+2i) = \chi(2+2i) = \chi(3+2i)$
7.	1	-1	-1	1	"conj. arg"		Line 4
8.	-1	-1	-1	—	—	—	$\chi(i) = \chi(1+i) = \chi(2+i)$
9.	-1	-1	1	-1	—	—	$\chi(2i) = \chi(1+2i) = \chi(2+2i)$
10.	-1	-1	1	1	—	—	$\chi(1-i) = \chi(2-i) = \chi(3-i)$
11.	-1	1	"conj. arg." Lines 8, 9, 10				

Hence $\Lambda_G^*(2,3) \leq \sqrt{18}$. Gaussian primes whose quadratic character has

$$\chi(i) = \chi(3+2i) = \chi(3-2i) = 1,$$

and

$$\chi(1+i) = \chi(2+i) = \chi(2-i) = \chi(3) = \chi(4+i) = \chi(4-i) = -1,$$

establish that $\sqrt{18}$ is the best possible value.

Exceptional primes are $1+i$, $2+i$, $2-i$, and 3.

TABLE D

	i	$1+i$	$2+i$	$2-i$	3	$3+2i$	$3-2i$	$4+i$	Consecutives
1.	1	1	—	—	—	—	—	—	$\chi(1-i)=\chi(1)=\chi(1-i)=1$
2.	1	—	—	—	1	—	—	—	$\chi(1)=\chi(2)=\chi(3)=1$
3.	1	−1	—	—	−1	1	—	—	$\chi(3+2i)=\chi(3+3i)=\chi(3+4i)=1$
4.	1	−1	—	—	−1	—	1	"conj. arg" Line 3	
5.	1	−1	1	1	−1	−1	−1	—	$\chi(5-i)=\chi(5)=\chi(5+i)=1$
6.	1	−1	−1	−1	−1	−1	−1	"conj. arg." Line 5	
7.	1	−1	1	−1	−1	−1	−1	1	$\chi(2+i)=\chi(3+i)=\chi(4+i)=1$
8.	1	−1	1	−1	−1	−1	−1	−1	$\chi(3-3i)=\chi(4-3i)=\chi(5-3i)=1$
9.	1	−1	−1	1	−1	−1	−1	"conj. arg." Lines 7 and 8	
10.	−1	1	—	−1	—	—	—	—	$\chi(1)=\chi(1+i)=\chi(1+2i)=1$
11.	−1	1	−1	—	"conj. arg." Line 10				
12.	−1	1	1	1	—	—	—	—	$\chi(1+i)=\chi(2+i)=\chi(3+i)=1$
13.	−1	−1	"conj. arg." Lines 10, 11, and 12						

Hence $\Lambda_G(2,3) \leq \sqrt{34}$. Gaussian primes whose quadratic character has

$$\chi(i) = \chi(2+i) = 1,$$

and

$$\begin{aligned}\chi(1+i) &= \chi(2-i) = \chi(3+2i) = \chi(3-2i) = \chi(4+i) \\ &= \chi(4-i) = \chi(5+2i) = \chi(5-2i) = -1,\end{aligned}$$

show $\sqrt{34}$ is the least value that works.

The exceptional primes are $1+i$, $2+i$, $2-i$, 3, $4+i$, and $4-i$.

In Tables E and F, the symbol χ is a cubic character, $w = (1+i\sqrt{3})/2$, and $\bar{w} = (-1-i\sqrt{3})/2$.

TABLE E

	$1+i$	$2+i$	Consecutives
1.	1		$\chi(1) = \chi(1+i)$
2.	w	w	$\chi(1+i) = \chi(2+i)$
3.	w	$\bar{w}$	$\chi(2) = \chi(2+i)$
4.	w	1	$\chi(2+i) = \chi(2+2i)$
5.	$\bar{w}$	"conj. arg." to Lines 2, 3, and 4	

Hence $\Lambda_G^*(3,2) \leq \sqrt{8}$. Gaussian primes whose cubic characters have $\chi(1+i) = w$ and $\chi(2+i) = 1$ show $\sqrt{8}$ is best possible.

The exceptional prime is $1+i$.

TABLE F

	$1+i$	$2+i$	$2-i$	3	$3+2i$	$3-2i$	$4+i$	$4-i$	Consecutives
1.	1	—	—	—	—	—	—	—	$\chi(1) = \chi(1+i) = 1$
2.	—	1	—	—	—	—	—	—	$\chi(2) = \chi(2+i) = 1$
3.	—	—	1	"conj. arg." Line 2					
4.	—	—	—	—	1	—	—	—	$\chi(2+2i) = \chi(3+2i) = 1$
5.	w	w	—	—	w	—	—	—	$\chi(11+2i) = \chi(11+3i) = 1$
6.	w	w	w	—	$\bar{w}$	—	—	—	$\chi(7+i) = \chi(8+i) = 1$
7.	w	w	$\bar{w}$	—	$\bar{w}$	—	—	—	$\chi(5) = \chi(5-i) = 1$
8.	w	$\bar{w}$	—	1	—	—	—	—	$\chi(3) = \chi(3-i) = 1$
9.	w	$\bar{w}$	—	w	w	—	—	—	$\chi(6+3i) = \chi(6+4i) = 1$
10.	w	—	$\bar{w}$	—	—	—	1	—	$\chi(3+i) = \chi(4+i) = 1$
11.	w	$\bar{w}$	—	—	—	—	—	1	$\chi(3-i) = \chi(4-i) = 1$
12.	w	—	$\bar{w}$	—	—	—	w	w	$\chi(8-2i) = \chi(9-2i) = 1$
13.	w	$\bar{w}$	—	w	—	—	—	$\bar{w}$	$\chi(5+3i) = \chi(6+3i) = 1$
14.	w	—	$\bar{w}$	w	—	—	$\bar{w}$	—	$\chi(5-3i) = \chi(6-3i) = 1$
15.	—	—	$\bar{w}$	—	w	—	—	—	$\chi(8) = \chi(8+i) = 1$
16.	w	—	$\bar{w}$	$\bar{w}$	$\bar{w}$	—	—	—	$\chi(12+8i) = \chi(12+9i) = 1$
17.	$\bar{w}$	"conj. arg." Lines 5–16							

Hence $\Lambda_G(3,2) \leq 15$. Gaussian primes whose cubic character has

$$\chi(1+i) = \chi(6+5i) = \chi(6-5i) = \chi(7+2i) = \chi(7-2i) = \chi(a+bi) = w$$

and

$$\begin{aligned}\chi(2+i) &= \chi(2-i) = \chi(3) = \chi(3+2i) = \chi(3-2i) = \chi(4+i)\\ &= \chi(4-i) = \chi(5+2i) = \chi(5-2i) = \chi(5+4i) = \chi(5-4i)\\ &= \chi(6+i) = \chi(6-i) = \chi(7) = \bar{w}\end{aligned}$$

shows that line 15 in Table F gives the best possible result.

Exceptional primes are $1+i$, $3+2i$, $3-2i$.

The proofs of G and H are identical and exhibit a method similar to the method of Lehmer and Lehmer [5] and Graham [3].

For $k = 2$ let M be arbitrary. Assign the characters values for all primes $a+bi$ whose absolute value is $\leq M$ as follows:

$$\begin{aligned}\chi(a+bi) &= 1 \quad \text{if} \quad a+bi \equiv -1, 0, \text{ or } 1 \pmod{2+i}\\ &= -1 \quad \text{if} \quad a+bi \equiv i \text{ or } -i \pmod{2+i}.\end{aligned}$$

Since $\{-1, 1, 0, i, -i\}$ is a complete residue system modulo $2+i$, all primes $a+bi$ whose absolute value is $\leq M$ are assigned. Since the set of Gaussian integers $H_1 = \{c+di\colon c+di \equiv -1, 0, \text{ or } i \pmod{2+i}\}$ is closed under multiplication and the set of Gaussian integers $H_2 = \{c+di\colon c+di \equiv i \text{ or } -i \pmod{2+i}\}$ has the property that the product of an even number of them is congruent to -1 or $1 \pmod{2+i}$, we have $\chi(c+di) = 1$ if $c+di$ is in H_1 and $\chi(c+di) = -1$ if $c+di$ is in H_2. Now any five consecutive Gaussian integers form a complete residue modulo $2+i$, so at most three consecutive Gaussian integers can have the same character value when $|c+di| \leq M$. Hence

$$\Lambda_G^*(2,m) \quad \text{and} \quad \Lambda_G(2,m) > M \quad \text{when} \quad m \geq 4.$$

For $k > 2$ let M be arbitrary. Assign the kth power character values as follows

$$\begin{aligned}\chi(1+i) &= \rho \qquad \rho \text{ a primitive } k\text{th root of unity},\\ \chi(a+bi) &= 1 \qquad \text{for } a+bi \text{ odd and } |a+bi| \leq M.\end{aligned}$$

Now each class of kth power non-residues contains only even Gaussian integers whose absolute value is $\leq M$ so does not even have two consecutive Gaussian integers in that range. The class of kth power residues contains all odd Gaussian integers whose absolute value is $\leq M$ and some multiples of $(1+i)^k$, $k \geq 3$. Since any four consecutive Gaussian integers is a subset of a complete residue system modulo $-2+2i = (1+i)^3$, at least one is divisible by $1+i$ but not by $(1+i)^3$. Hence at least one fails to be a residue when the consecutives have absolute value $\leq M$. So at most three consecutive Gaussian integers whose absolute values are $\leq M$ can have the same character value. Therefore

$$\Lambda_G^*(k, m) \quad \text{and} \quad \Lambda_G(k, m) > M \quad \text{when} \quad m \geq 4 \quad \text{and} \quad k \geq 3.$$

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